Limits and Their Properties

1.2 Finding Limits Graphically and Numerically

Objectives

- Estimate a limit using a numerical or graphical approach.
- Learn different ways that a limit can fail to exist.
- Study and use a formal definition of limit.

An Introduction to Limits

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To sketch the graph of the function

$$f(x) = \frac{x^3 - 1}{x - 1}$$

for values other than $x = 1$, you can use standard curve-sketching techniques. At $x = 1$, however, it is not clear what to expect.

An Introduction to Limits

To get an idea of the behavior of the graph of f near $x = 1$, you can use two sets of x-values–one set that approaches 1 from the left and one set that approaches 1 from the right, as shown in the table.

x approaches 1 from the left.

x approaches 1 from the right.

x	0.75	0.9	0.99	0.999	1	1.001	1.01	1.1	1.25
$f(x)$	2.313	2.710	2.970	2.997	?	3.003	3.030	3.310	3.813

$f(x)$ approaches 3.

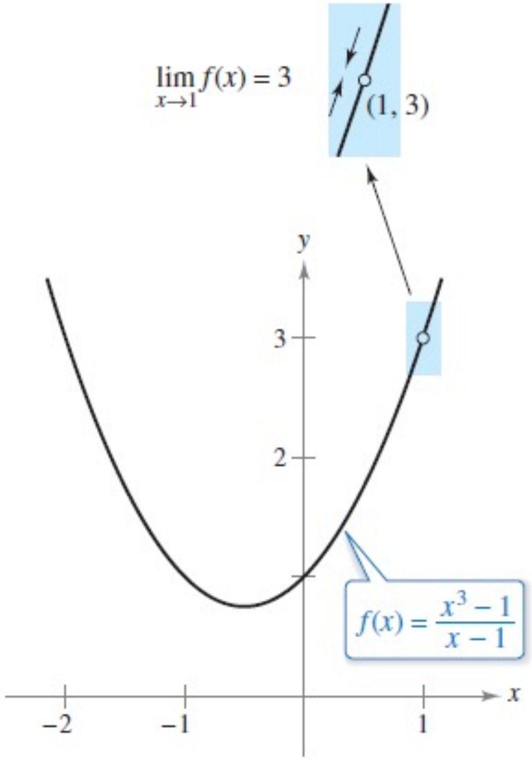
$f(x)$ approaches 3.

An Introduction to Limits

The graph of *f* is a parabola that has a hole at the point (1, 3), as shown in the Figure 1.5.

Although *x* cannot equal 1, you can move arbitrarily close to 1, and as a result *f*(*x*) moves arbitrarily close to 3.

Using limit notation, you can write



The limit of $f(x)$ as x approaches 1 is 3.

Figure 1.5

$$\lim_{x \to 1} f(x) = 3.$$

This is read as “the limit of *f*(*x*) as *x* approaches 1 is 3.”

An Introduction to Limits

This discussion leads to an informal definition of limit.

If $f(x)$ becomes arbitrarily close to a single number L as x approaches c from either side, the **limit** of $f(x)$ as x approaches c is L.

This limit is written as

$$\lim_{x \to c} f(x) = L.$$

Example 1 – *Estimating a Limit Numerically*

Evaluate the function $f(x) = x/(\sqrt{x+1} - 1)$ at several *x*-values near 0 and use the results to estimate the limit

$$\lim_{x \to 0} \frac{x}{\sqrt{x+1} - 1}.$$

Example 1 – *Solution*

The table lists the values of $f(x)$ for several x-values near 0.

x approaches 0 from the left. x approaches 0 from the right.

x	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01
$f(x)$	1.99499	1.99950	1.99995	?	2.00005	2.00050	2.00499

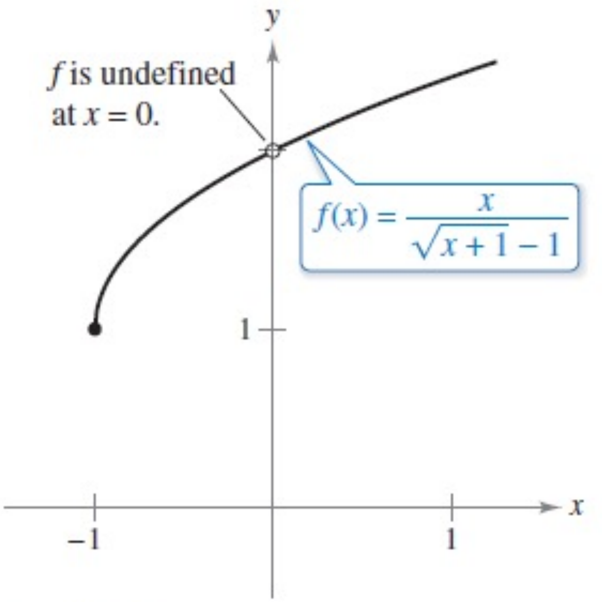
$f(x)$ approaches 2. $f(x)$ approaches 2.

From the results shown in the table, you can estimate the limit to be 2.

Example 1 – *Solution*

cont'd

This limit is reinforced by the graph of *f* shown in Figure 1.6.



The limit of $f(x)$ as x approaches 0 is 2.

Figure 1.6

Limits That Fail to Exist

Example 3 – *Different Right and Left Behavior*

Show that the limit $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

Solution:

Consider the graph of the function

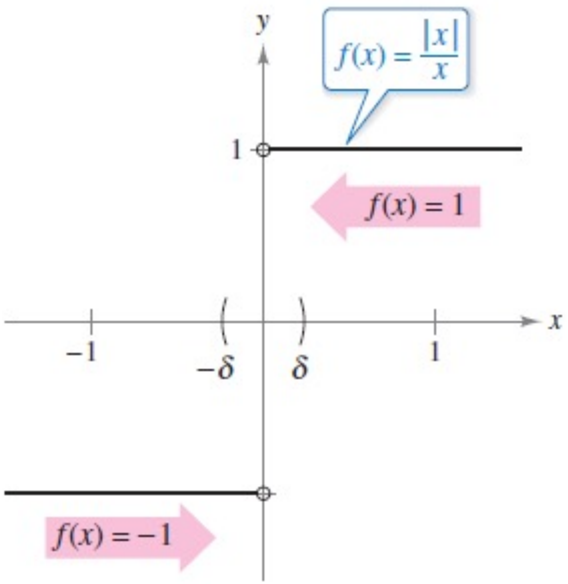
$$f(x) = \frac{|x|}{x}.$$

In Figure 1.8 and from the definition of absolute value,

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$ Definition of absolute value

you can see that

$$\frac{|x|}{x} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$



$\lim_{x \to 0} f(x)$ does not exist.

Figure 1.8

Example 3 – *Solution*

cont'd

So, no matter how close x gets to 0, there will be both positive and negative x-values that yield $f(x) = 1$ or $f(x) = -1$.

Specifically, if δ (the lowercase Greek letter delta) is a positive number, then for x-values satisfying the inequality $0 < |x| < \delta$, you can classify the values of $|x|/x$ as -1 or 1 on the intervals

$$(-\delta, 0) \quad \text{or} \quad (0, \delta).$$

Negative x-values yield $|x|/x = -1$.

Positive x-values yield $|x|/x = 1$.

Example 3 – *Solution*

cont'd

Because $|x|/x$ approaches a different number from the right side of 0 than it approaches from the left side, the limit $\lim_{x \to 0} (|x|/x)$ does not exist.

Limits That Fail to Exist

Common Types of Behavior Associated with Nonexistence of a Limit

1. $f(x)$ approaches a different number from the right side of c than it approaches from the left side.
2. $f(x)$ increases or decreases without bound as x approaches c.
3. $f(x)$ oscillates between two fixed values as x approaches c.

A Formal Definition of Limit

A Formal Definition of Limit

Let's take another look at the informal definition of limit. If *f*(*x*) becomes arbitrarily close to a single number *L* as *x* approaches *c* from either side, then the limit of *f*(*x*) as *x* approaches *c* is *L*, is written as

$$\lim_{x \to c} f(x) = L.$$

At first glance, this definition looks fairly technical. Even so, it is informal because exact meanings have not yet been given to the two phrases

"*f*(*x*) becomes arbitrarily close to *L*"

and

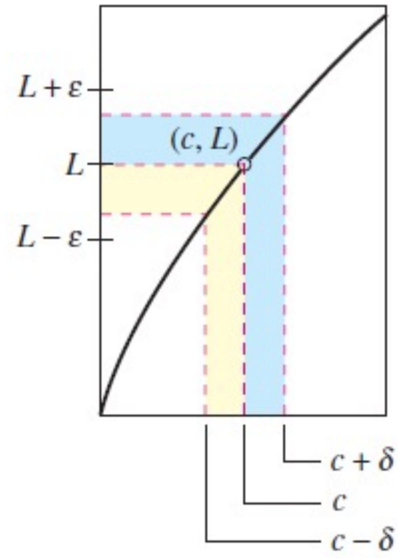
"*x* approaches *c*."

A Formal Definition of Limit

The first person to assign mathematically rigorous meanings to these two phrases was Augustin-Louis Cauchy. His ε-δ **definition of limit** is the standard used today.

In Figure 1.12, let ε (the lower case Greek letter epsilon) represent a (small) positive number.

Then the phrase "*f*(*x*) becomes arbitrarily close to *L*" means that *f*(*x*) lies in the interval $(L - \varepsilon, L + \varepsilon)$.



The ε-δ definition of the limit of $f(x)$ as x approaches c

Figure 1.12

A Formal Definition of Limit

Using absolute value, you can write this as

$$|f(x) - L| < \varepsilon.$$

Similarly, the phrase *"x approaches c"* means that there exists a positive number δ such that *x* lies in either the interval $(c - \delta, c)$ or the interval $(c, c + \delta)$.

This fact can be concisely expressed by the double inequality

$$0 < |x - c| < \delta.$$

A Formal Definition of Limit

The first inequality

$$0 < |x - c| \qquad \text{The distance between } x \text{ and } c \text{ is more than } 0.$$

expresses the fact that $x \neq c$. The second inequality

$$|x - c| < \delta \qquad x \text{ is within } \delta \text{ units of } c.$$

says that *x* is within a distance δ of *c.*

A Formal Definition of Limit

Definition of Limit

Let f be a function defined on an open interval containing c (except possibly at c), and let L be a real number. The statement

$$\lim_{x \to c} f(x) = L$$

means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if

$$0 < |x - c| < \delta$$

then

$$|f(x) - L| < \varepsilon.$$

Example 6 – *Finding a δ for a Given ε*

Given the limit

$$\lim_{x \to 3} (2x - 5) = 1$$

find δ such that $|(2x - 5) - 1| < 0.01$ whenever $0 < |x - 3| < \delta.$

Solution:

In this problem, you are working with a given value of ε–namely, $\varepsilon = 0.01$.

To find an appropriate δ, try to establish a connection between the absolute values

$$|(2x - 5) - 1| \quad \text{and} \quad |x - 3|.$$

Example 6 – *Solution* cont'd

Notice that

$$|(2x - 5) - 1| = |2x - 6| = 2|x - 3|.$$

Because the inequality $|(2x - 5) - 1| < 0.01$ is equivalent to $2|x - 3| < 0.01$, you can choose

$$\delta = \tfrac{1}{2}(0.01) = 0.005.$$

This choice works because

$$0 < |x - 3| < 0.005$$

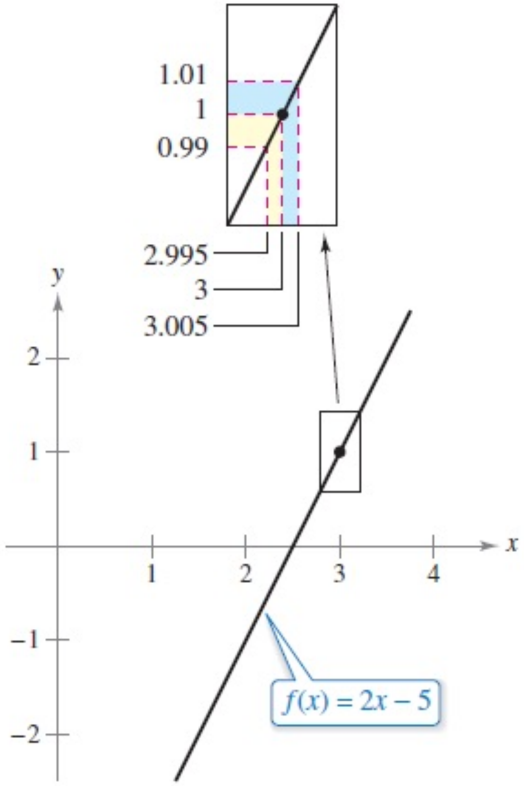
implies that

$$|(2x - 5) - 1| = 2|x - 3| < 2(0.005) = 0.01.$$

Example 6 – *Solution*

cont'd

As you can see in Figure 1.13, for *x*-values within 0.005 of 3 ($x \neq 3$), the values of $f(x)$ are within 0.01 of 1.



The limit of $f(x)$ as x approaches 3 is 1.

Figure 1.13